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Equivalence Structures and Their Automorphisms

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An equivalence structure (X, E) is a set E of equivalence relations on a set X such that any two distinct elements of X are related in one, and only one, relation of E. We give bounds for the size of the set E under certain conditions on E. We show that every group can occur as the automorphism group of an equivalence structure.

0. INTRODUCTION

A society has a series of dinners each year such that every member attends each dinner. The rules of the society state that any two of its members have to dine at a table together exactly once every year. If at the first dinner in the year the members occupy the tables in some constellation, how many dinners will the society have to hold that year to fulfill the rules?

One can also ask the question what the minimal number of dinners is that the society has to hold in order to meet the requirements, if the first dinner has not yet been held. Trivially, they can meet once and sit all together at one table. But what happens if the rules of the society forbid that all members meet at one table?

We shall now formulate the problems in a different way. If X is a set and E is a set of equivalence relations on X then we shall call the pair (X, E) an *equivalence system.* If we have the further property that for $x, y \in X$ with $x \neq y$ there exists one, and only one, $e \in E$ such that x and y are in the relation e (which we abbreviate as x e y) then we say that (X, E) is an *equivalence structure.*

We first note that if e is an equivalence relation on a set X then there always is an equivalence structure (X, E) with $e \in E$. Namely, for every two element subset $\{x, y\}$ of X such that we do not have x e y we define an equivalence relation $f(\{x, y\})$ as having equivalence classes $\{x, y\}$ and one-element classes otherwise, and then let E consist of e and all these relations $f(\{x, y\})$. Therefore we can define dim(e), the *dimension* of e, to be the minimal cardinal number such that there exists an equivalence structure (X, E) with $e \in E$ and $\dim(e) = |E|$. We say than an equivalence structure (X, E) is trivial if E contains the relation $a = X \times X$ (that is, the relation in which every two elements of X are related). Clearly, (X, E) is trivial if and only if $E = \{a\}$ or e consists of the relation a and the equality relation. If n is any cardinal number we define by $R(n)$ the minimal cardinal such that there exists a non-trivial equivalence structure (X, E) with $|X| = n$ and $|E| = R(n)$.

Now we note that there is a well-known correspondence between partitions and equivalence relations (viz the blocks of a partition being the equivalence classes of an equivalence relation). From this we see that the first combinatorial problem stated above is the determination of dim(e) when the relation e is given, and the second problem is the determination of $R(n)$.

1. FINITE EQUIVALENCE STRUCTURES

Let e be an equivalence relation on a finite set X with $|X| = n$. We want to give an upper bound for dim(e). We first note that there is an equivalence relation e with $\dim(e) = n$. For, let $X = \{0, 1, \ldots, n-1\}$, and let e be the relation with classes $\{0\}$ and $\{1, 2, \ldots, n-1\}$.

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Then clearly the best we can get for E is $E = \{\mathrm{e}, \mathrm{f}_i | 1 \leqslant i \leqslant n - 1\}$, where f_i is the relation whose classes are $\{0, i\}$ and one-element classes otherwise. But this is the worst case indeed.

THEOREM 1.1. *If* e *is an equivalence relation on a finite set* X *with* $|X| = n$ *then* $\dim(\mathrm{e}) \leqslant n$.

PROOF. First we make some preliminary remarks. If k is a positive integer and x is an integer we denote by $r_k(x)$ the unique integer with the properties $x \equiv r_k(x) \pmod k$ and $0 \leqslant r_k(x) < k$ (the remainder of x when divided by k). It is easy to see that if k is an odd positive integer then for each j with $0 \leqslant j < k$ a partition P_j of $\{0, 1, \ldots, k - 1\}$ is given by the blocks $\{j\}$, $\{r_k(j + i), r_k(j - i)\}$ for $1 \leqslant i \leqslant (k - 1)/2$, and that each two-element subset of $\{0, 1, \ldots, k - 1\}$ is a block in exactly one of these partitions. Also, if n is of the form $n = 2k$ with k odd, there are $n - 1$ partitions P_j (for $0 \leqslant j \leqslant k - 1$) and P_j' (for $1 \leqslant j \leqslant k - 1$) of $\{0, 1, \ldots, n - 1\}$, where P_j has the blocks $\{j, k + j\}$, $\{r_k(j + i), r_k(j - 1)\}$, $\{k + r_k(j + i), k + r_k(j - i)\}$ for $1 \leqslant i \leqslant (k - 1)/2$, and where P_j' has the blocks $\{i, k + r_k(i + j)\}$ for $0 \leqslant i \leqslant k - 1$. Again, each two-element subset of $\{0, 1, \ldots, n - 1\}$ is a block in exactly one of those partitions.

Furthermore, if $n = 2^m k$ with k odd and $m \geqslant 1$, we can see by induction on m that there are $n - 1$ partitions of $\{0, 1, \ldots, n - 1\}$ in two-element subsets such that each two-element subset of $\{0, 1, \ldots, n - 1\}$ is a block of exactly one of these partitions. For $m = 1$ we have shown it above. So we can assume that there are $2^{m-1}k$ partitions P_j of $\{0, 1, \ldots, 2^{m-1}k - 1\}$ and P_j' of $\{2^{m-1}k, \ldots, 2^m k - 1\}$ with the given properties. We now get partitions $P_j'' = P_j \cup P_j'$ of $\{0, 1, \ldots, n - 1\}$ for $1 \leqslant j \leqslant 2^{m-1}k - 1$. We define partitions P_j''' of $\{0, 1, \ldots, n - 1\}$ for $0 \leqslant j \leqslant 2^{m-1}k - 1$ as having blocks $\{i, 2^{m-1}k + r_{2^{m-1}k}(i + j)\}$ for $0 \leqslant i \leqslant 2^{m-1}k - 1$. Then each two-element subset of $\{0, 1, \ldots, n - 1\}$ is a block in exactly one of the $n - 1$ partitions P_j'' or P_j'''.

Now we can prove the theorem. First let us assume that n is even. Without loss of generality we can assume that $X = \{0, 1, \ldots, n - 1\}$. Now we have seen that there are partitions P_j of X for $1 \leqslant j \leqslant n - 1$ into two-element subsets such that each two-element subset of X is a block in exactly one of the partitions. We then define equivalence relations f_j for $1 \leqslant j \leqslant n - 1$ by $x \mathrm{f}_j\, y$ for $x \neq y$ if and only if $\{x, y\}$ is a block of P_j and we do not have $x \,\mathrm{e}\, y$. Then it is clear that (X, E) is an equivalence structure where $E = \{\mathrm{e}\} \cup \{\mathrm{f}_j | 1 \leqslant j \leqslant n - 1\}$ with $|E| \leqslant n$.

Finally we assume that n is odd. We first define an equivalence relation f on X in the following way. Choose one element from each equivalence class of e and let the set of these elements be the first equivalence class of f. Then choose one of the remaining elements from each class of e (where there are any left) to get the second class of f, and so forth. Note that we can write the elements of X in a sequence $x_0, x_2, x_{n-2}, x_4, x_{n-4}, x_6, x_{n-6}, \ldots, x_{n-1}, x_1$ such that each element of this sequence is in relation e or f with its successor. (To do this, list the equivalence classes of e in some order of non-decreasing cardinality, then start the sequence with all the elements from the first class. By construction of f and the way the classes of e are listed, there exists an element of the second class of e which is in relation f to the last element of the sequence so far constructed. Take this element as the next element of the sequence, and then all other elements of the second class. Continue this process untill all elements of X are listed.)

So (by identifying x_i with i) we can assume that $X = \{0, 1, \ldots, n - 1\}$ and that any pair of elements $(r_n(i), r_n(-i))$ and $(r_n(1 + i), r_n(1 - i))$ is in relation e or f. Now we define relations $\mathrm{f}_2, \ldots, \mathrm{f}_{n-1}$ by $x \,\mathrm{f}_j\, y$ for $x \neq y$ if and only if $\{x, y\}$ is a block of the partition $P_j = \{\{j\}, \{r_n(j + 1), r_n(j - i)\} | 1 \leqslant i \leqslant (n - 1)/2\}$ and neither $x \,\mathrm{e}\, y$ nor $x \,\mathrm{f}\, y$. Then it is not hard to see that for $E = \{\mathrm{e}, \mathrm{f}\} \cup \{\mathrm{f}_j | 2 \leqslant j \leqslant n - 1\}$ we have an equivalence structure (X, E) with $|E| \leqslant n$. This concludes the proof of Theorem 1.1.

Having established an upper bound for dim(e) we now want to get a lower bound for $R(n)$. To do this we need a few lemmas.

LEMMA 1.2. *Let n, k be positive integers. If there exists a number m with $1 < m < n$ and $k \geqslant m + 1$ and $k \geqslant (n - 1)/(m - 1)$ then $k \leqslant \sqrt{n} + 1$.*

PROOF. We prove the lemma by contradiction. Assume that $k < \sqrt{n} + 1$ and that m satisfies the stated conditions. It then follows that $m + 1 < \sqrt{n} + 1$ and $(n - 1)/(m - 1) < \sqrt{n} + 1$. From the first inequality we get $m < \sqrt{n}$. The second inequality then implies $n - 1 < (\sqrt{n} + 1)(m - 1) < (\sqrt{n} + 1)(\sqrt{n} - 1) = n - 1$ which is a contradiction. So we have $k \geqslant \sqrt{n} + 1$.

LEMMA 1.3. *Let $n, m, p_1, \ldots, p_r$ be non-negative integers with $n = p_1 + \cdots + p_r$ and $m \geqslant p_1 \geqslant \cdots \geqslant p_r$. Then $\Sigma_{i=1}^{r} p_i(p_i - 1) \leqslant n(m - 1)$.*

PROOF. $\Sigma_{i=1}^{r} p_i(p_i - 1) \leqslant \Sigma_{i=1}^{r} p_i(m - 1) = (m - 1) \Sigma_{i=1}^{r} p_i = n(m - 1)$

THEOREM 1.4. *If n is a positive integer then $R(n) \geqslant \sqrt{n} + 1$.*

PROOF. Let (X, E) be a non-trivial equivalence structure with $X = \{1, 2, \ldots, n\}$ and $E = \{\mathrm{e}_1, \ldots, \mathrm{e}_k\}$. Let m be the maximal cardinality of any equivalence class of any of the relations $\mathrm{e}_1, \ldots, \mathrm{e}_k$. Say, the relation e_i has equivalence classes $C_{i1}, \ldots, C_{i,s(i)}$ with $|C_{i1}| = m$. As (X, E) is non-trivial, we must have $s(i) \geqslant 2$, so there exists $x \in C_{i2}$. Now for each $y \in C_{i1}$ there must exist a relation $\mathrm{e} \in E \backslash \{\mathrm{e}_i\}$ with $x\ \mathrm{e}\ y$, and if $y \neq y' \in C_{i1}$ and $x\ \mathrm{e}'\ y'$ then $\mathrm{e} \neq \mathrm{e}'$ (as otherwise $y\ \mathrm{e}\ y'$, which would be a contradiction as we also have $y\ \mathrm{e}_i\ y'$). So we get $k \geqslant m + 1$.

Now for each two-element subset $\{x, y\}$ of $\{1, 2, \ldots, n\}$ there exists exactly one equivalence relation $\mathrm{e} \in E$ containing an equivalence class C with $\{x, y\} \subseteq C$. The number of two-element subsets thus occuring as a subset of an equivalence class of the equivalence relation e_j with classes $C_{j1}, \ldots, C_{j,s(j)}$ is $\Sigma_{t=1}^{s(j)} |C_{jt}| \cdot (|C_{jt}| - 1)/2$ which by Lemma 1.3 is less than or equal to $n(m - 1)/2$. Therefore to get all two-element subsets of $\{1, 2, \ldots, n\}$ we must have $kn(m - 1)/2 \geqslant n(n - 1)/2$, and hence $k \geqslant (n - 1)/(m - 1)$. By Lemma 1.2 it now follows that $k \geqslant \sqrt{n} + 1$. As this is true for every non-trivial equivalence structure (X, E), we must have $R(n) \geqslant \sqrt{n} + 1$.

Note that the first part of the proof also shows that for any equivalence relation e on a set X we get $\dim(\mathrm{e}) \geqslant m + 1$ where m is the maximal cardinal of an equivalence class of e. Now we also want to see if the inequality $R(n) \geqslant \sqrt{n} + 1$ is sharp, that is, if we can ever have equality. Clearly a necessary condition for this is that n is a perfect square.

THEOREM 1.5. *Let E be a set of equivalence relations on a set X with $|X| = n^2 > 1$. Then the following are equivalent.*

(i) The pair (X, E) is a non-trivial equivalence structure with $|E| = n + 1$.

(ii) The equivalence classes of the relations $\mathrm{e} \in E$ are the lines of an affine plane on X, with two lines being parallel if they are classes for the same relation e.

PROOF. We first assume (ii). Let E be the set of parallel classes of an affine plane X such that $x\ \mathrm{e}\ y$ for $x \neq y$ if, and only if, the line through x and y lies in e. Then clearly $|E| = n + 1$, and (X, E) is an equivalence structure as for every $x, y \in X$ with $x \neq y$ there exists exactly one line through x and y and hence exactly one relation e with $x\ \mathrm{e}\ y$. Also E is non-trivial as $n^2 > 1$. So we have (i).

Now assume (i). Let (X, E) be a non-trivial equivalence structure with $|E| = n + 1$. Let m be the maximal cardinality of any equivalence class of any one of the relations $\mathrm{e} \in E$. As in the proof of Theorem 1.4, it follows that $(n + 1) . n^2 . (m - 1)/2 \geqslant n^2 . (n^2 - 1)/2$, and hence $m - 1 \geqslant n - 1$. But in Theorem 1.4 we have also seen that we must have $n + 1 \geqslant m + 1$. So we get $m = n$. Now the number of two-element subsets of X which are subsets of some equivalence class of a relation $\mathrm{e} \in E$ is at most $n^2(n - 1)/2$ for each $\mathrm{e} \in E$. Therefore to get the total of $n^2(n^2 - 1)/2$ such sets this number must, in fact, be equal to $n^2(n - 1)/2$ for each $\mathrm{e} \in E$. But by Lemma 1.3 it is easily seen that this is only possible if all equivalence classes of e have the same cardinality n. As we also know that any two distinct elements of X are related in exactly one relation $\mathrm{e} \in E$ and then lie in exactly one equivalence class, it follows by 3.2.4(b) in [3] that the equivalence classes form the lines of an affine plane of order n, and we have (ii).

We know that $R(n^2) = n + 1$ if and only if there exists an affine plane of order n. Also it is clear that the function $n \mapsto R(n)$ is non-decreasing, as it is easy to see that if (X, E) is a non-trivial equivalence structure then for any n with $3 \leqslant n \leqslant |X|$ there exists a subset Y of X with $|Y| = n$ such that if F is the set of relations in E restricted to $Y \times Y$ then (Y, F) is a non-trivial equivalence structure with $|F| \leqslant |E|$. In particular, using Theorem 1.5 and the classical affine planes it follows that $R(n) \leqslant q + 1$ where q is any prime power with $q \geqslant \sqrt{n}$. From this if follows that $R(n)$ can be bounded above by $R(n) < \frac{7}{6}(\sqrt{n} + 1) + 1$. This inequality holds because for every natural number n there exists a prime power q such that $\sqrt{n} \leqslant q < \frac{7}{6}(\sqrt{n} + 1)$. This follows by checking explicitly for small n, and by a suitable generalization of Bertrand's postulate (see, for example, [10] and the references given there). Of course, for large n this bound can be improved further.

2. Infinite Equivalence Structures

Whereas the determination of dim(e) and $R(n)$ in the finite case is complicated (the problem of determining $R(n^2)$ includes the question of the existence of an affine plane of order n) the corresponding questions for infinite sets X can be completely answered.

Theorem 2.1. *Let X be an infinite set. If (X, E) is a non-trivial equivalence structure then* $|E| = |X|$.

Proof. We first show that $|E| \leqslant |X|$. Let E' be E without the equality relation (which may or may not lie in E). For each relation $\mathrm{e} \in E'$ we choose two elements $x, y \in X$ with $x \neq y$ and $x \mathrel{\mathrm{e}} y$. So we get an injection from E' into the set of two-element subsets of X, which has the same cardinality as X. So we have $|E| \leqslant |X|$.

To show the opposite inequality, consider the cardinalities of all equivalence classes of all the relations $\mathrm{e} \in E$. We first assume that $|X|$ is the smallest cardinal which is greater than or equal to $|C|$ for all equivalence classes C of all the relations e. As (X, E) is non-trivial, for each class C there exists $x \in X \backslash C$, and in order for each pair $\{x, y\}$ with $y \in C$ to be contained in some class of a relation, we must have $|E| \geqslant |C|$ for all equivalence classes C. So we have $|E| \geqslant |X|$.

Now suppose there is a cardinal n with $n < |X|$ and such that $|C| \leqslant n$ for all equivalence classes C. Let $x \in X$. Then the number of elements related to x in one relation e is at most n. So the number of elements related to x in any of the relations of E is at most $n . |E|$ (see [7], V. 7, thm. 8). As every element of X must be related to x in some relation, we must have $n . |E| \geqslant |X|$, and hence $|E| \geqslant |X|$.

3. Automorphisms of Equivalence Systems

If X is a set we write Sym(X) for the group of all permutations of X. If (X, E) is an equivalence system, then let $\mathrm{Aut}(X, E) = \{g \in \mathrm{Sym}(X) |$ For all $\omega, \omega' \in X$ and for all $\mathrm{e} \in E$

we have ω e ω' if and only if (ωg) e $(\omega' g)$. Clearly $\mathrm{Aut}(X, E)$ is a subgroup of $\mathrm{Sym}(X)$, which we shall call the *automorphism group* of (X, E). Note that if one imposes restrictions on (X, E) one can show that $\mathrm{Aut}(X, E)$ is a group of fairly restricted type, e.g. a generalized wreath product of symmetric groups. (theorem B in [1]). We shall now see that every group can occur as $\mathrm{Aut}(X, E)$ for some (X, E).

THEOREM 3.1. *Let G be any group. Then there exists an equivalence system (X, E) such that G is isomorphic to* $\mathrm{Aut}(X, E)$. *We can take X, E such that* $|X| = 2|G| + 1$ *and* $|E| = |G| + 1$.

PROOF. We define $X := \{0, 1\} \times G \cup \{\infty\}$. We define the equivalence relation f on X as having the two classes $\{(0, g) \mid g \in G\} \cup \{\infty\}$ and $\{(1, g) \mid g \in G\}$. For each $h \in G$ we define an equivalence relation e_h as having classes $\{\infty\}$ and $\{(0, g), (1, hg)\}$ for all $g \in G$. Now $E := \{f\} \cup \{e_h \mid h \in G\}$, and we claim that $G \cong \mathrm{Aut}(X, E)$.

First note that we get a monomorphism $\phi : G \to \mathrm{Aut}(X, E)$ in the following way. If $k \in G$ then $\infty . (k\phi) = \infty$ and $(x, g)(k\phi) = (x, gk)$ for $x \in \{0, 1\}$, $g \in G$. So it remains to show that every element of $\mathrm{Aut}(X, E)$ is of the form $k\phi$ for some $k \in G$.

Let $\alpha \in \mathrm{Aut}(X, E)$. First note that all classes of e_1 except $\{\infty\}$ have 2 elements. As α permutes the classes we therefore must have $\infty\alpha = \infty$. As α leaves the relation f invariant we therefore get that α leaves each of the sets $\{(0, g) \mid g \in G\}$ and $\{(1, g) \mid g \in G\}$ setwise invariant. So there exists $k \in G$ such that $(0, 1)\alpha = (0, k)$. Now as $(1, g)$ is the unique other element related to $(0, 1)$ in the relation e_g and as $(1, gk)$ is the unique other element related to $(0, k)$ in the relation e_g, we therefore (as α leaves e_g invariant) must have $(1, g)\alpha = (1, gk)$ for all $g \in G$. In particular, we have $(1, 1)\alpha = (1, k)$. Similarly, as $(0, g)$ is the unique other element related to $(1, 1)$ in the relation $e_{g^{-1}}$ and $(0, gk)$ is the unique other element related to $(1, k)$ in the relation $e_{g^{-1}}$ it follows that $(0, g)\alpha = (0, gk)$ for all $g \in G$. So we have $\alpha = k\phi$, which completes the proof.

It is not hard to see that if S is any set of generators of G, it suffices to take $E = \{f\} \cup \{e_h, e_{h^{-1}} \mid h \in S\}$, so that in Theorem 3.1 we can get $|E| = 2m(G) + 1$ where $m(G)$ is the minimal cardinal of a set of generators of G. Modifying the construction of E we can get (X, E) to be an equivalence structure.

COROLLARY 3.2. *Let G be any group. Then there exists an equivalence structure (X, E) such that $G \cong$* $\mathrm{Aut}(X, E)$. *We can take* $|X| = 2|G| + 1$ *and* $|E| = |G| + 2$.

PROOF. As above, $X := \{0, 1\} \times G \cup \{\infty\}$. Let f_0 be the equivalence relation with classes $\{\infty\} \cup \{(0, g) \mid g \in G\}$ and $\{(1, g)\}$ for all $g \in G$. Similarly, let f_1 be the equivalence relation with classes $\{\infty\} \cup \{(1, g) \mid g \in G\}$ and $\{(0, g)\}$ for all $g \in G$. For each $h \in G$ we define the relation e_h as in the proof of Theorem 3.1. Then, if $E = \{f_0, f_1\} \cup \{e_h \mid h \in G\}$ it is easy to see that (X, E) is an equivalence structure, and as in Theorem 3.1 one shows that $G \cong \mathrm{Aut}(X, E)$.

This shows that equivalence systems and equivalence structures have the same kind of generality as other mathematical structures which also have the property that every group can occur as the full automorphism group of such a structure, for example partially ordered sets [2, 6], graphs [4, 5], projective planes [8] or Steiner triple systems [9].

4. SEMI-AUTOMORPHISMS OF EQUIVALENCE STRUCTURES

For equivalence structures, the concept of an automorphism as given in the preceding section can be considerably weakened. This is motivated by the equivalence structures in Theorem 1.5. The automorphisms of such an equivalence structure correspond to those collineations of the corresponding affine plane which map lines onto lines of the same parallel class. We now introduce semi-automorphisms of the equivalence structure which will correspond to all collineations of the affine plane.

Let (X, E) be an equivalence structure. We shall call a permutation $g \in \mathrm{Sym}(X)$ a *semi-automorphism* of (X, E) if for all $x, y, z, u \in X$ with $x \neq y$ the following holds. When e is the relation in E such that x e y and f is the relation in E such that (xg) f (yg) then z e u if and only if (zg) f (ug). Let $\mathrm{Saut}(X, E)$ be the set of all semi-automorphisms of (X, E).

LEMMA 4.1. *$\mathrm{Saut}(X, E)$ is a subgroup of $\mathrm{Sym}(X)$, and $\mathrm{Aut}(X, E)$ is a normal subgroup of $\mathrm{Saut}(X, E)$.*

PROOF. Let $g \in \mathrm{Saut}(X, E)$. Let $x, y, z, u \in X$ with $x \neq y$, x e y, (xg^{-1}) f (yg^{-1}). Now note that x e y is the same as $(xg^{-1})g$ e $(yg^{-1})g$. So as $g \in \mathrm{Saut}(X, E)$ it follows that (zg^{-1}) f (ug^{-1}) if and only if $(zg^{-1})g$ e $(ug^{-1})g$, which is the same as z e u. So $g^{-1} \in \mathrm{Saut}(X, E)$.

Now let $g, h \in \mathrm{Saut}(X, E)$. Let $x, y, z, u \in X$ with $x \neq y$, x e y, (xgh) f (ygh). Now let f' be the unique element of E with (xg) f$'$ (yg). Then z e u if and only if (zg) f$'$ (ug) as $g \in \mathrm{Saut}(X, E)$. And (zg) f$'$ (ug) if and only if $(zg)h$ f $(ug)h$, as $h \in \mathrm{Saut}(X, E)$. Hence we have z e u if and only if (zgh) f (ugh), and hence $gh \in \mathrm{Saut}(X, E)$.

Finally, let $g \in \mathrm{Saut}(X, E)$, $h \in \mathrm{Aut}(X, E)$. Let $x, y \in X$, x e y. We have to show that $(xg^{-1}hg)$ e $(yg^{-1}hg)$. We can assume that $x \neq y$. So let f be the unique element of E with (xg^{-1}) f (yg^{-1}). As $h \in \mathrm{Aut}(X, E)$, we have $(xg^{-1}h)$ f $(yg^{-1}h)$. As we also have $(xg^{-1})g$ e $(yg^{-1})g$ it follows that $(xg^{-1}h)g$ e $(yg^{-1}h)g$. Hence $g^{-1}hg \in \mathrm{Aut}(X, E)$.

EXAMPLE 4.2. For every ordinal $\alpha > 2$ we construct an equivalence structure $(S(\alpha), E(\alpha))$ with $\mathrm{Saut}(S(\alpha), E(\alpha)) = \{1\}$ which we shall need later on.

Let $\alpha > 2$ be an ordinal. We define $S(\alpha) = \{(\beta, \gamma) | \beta, \gamma \text{ ordinals}, \gamma \leqslant \beta < \alpha\}$. Let s be the equivalence relation on $S(\alpha)$ having the classes $\{(\beta, \gamma) | \gamma \text{ ordinal with } \gamma \leqslant \beta\}$ for all $\beta < \alpha$, that is, classes $\{(0, 0)\}, \{(1, 0), (1, 1)\}, \{(2, 0), (2, 1), (2, 2)\}, \ldots$. For each ordinal $\gamma < \alpha$ we define an equivalence relation t_γ as having the class $\{(\beta, \gamma) | \beta \text{ ordinal with } \gamma \leqslant \beta < \alpha\}$ and one-element classes otherwise. If $\alpha = \phi + 2$ for some ordinal ϕ then we define an equivalence relation r as having classe $\{(0, 0), (\phi + 1, \phi\}, \{(\phi, \phi), (\phi + 1, \phi + 1)\}$ and one-element classes otherwise. Finally, for each two-element subset $\{x, y\}$ of $S(\alpha)$ such that x, y are not related in any of the relations r, s or t_γ we define a relation $u(\{x, y\})$ as having the class $\{x, y\}$ and one-element classes otherwise.

Let $E(\alpha)$ be the set consisting of all the relations s, t_γ, $u(\{x, y\})$ and possibly r defined above. Then clearly $(S(\alpha), E(\alpha))$ is an equivalence structure, and we want to show that $G := \mathrm{Saut}(S(\alpha), E(\alpha)) = \{1\}$. This is easy for $\alpha = 3$. For $\alpha > 3$ we shall prove it by showing by (possibly transfinite) induction on β that all elements (β, γ) with $\gamma \leqslant \beta$ are left invariant under G. First we consider $\beta = 0$. As s is the only relation in $S(\alpha)$ which has a one-element class and at least two classes with more than one element, each element of G must permute the classes of s. But as $\{(0, 0)\}$ is the only class of s consisting of one element only, it follows that $(0, 0)$ must be fixed by G.

Now let $\beta > 0$. By the induction hypothesis, we have (γ, δ) fixed by G whenever $\beta > \gamma \geqslant \delta$. Let us first consider the case where there does not exist ϕ such that $\beta = \phi + 1$ and $\alpha = \phi + 2$. Let $\delta < \beta$. Then, as (δ, δ) is fixed, and $\{(\varepsilon, \delta) | \varepsilon \text{ ordinal with } \delta \leqslant \varepsilon < \alpha\}$ is the unique equivalence class of any relation in $E \backslash \{s\}$ with more than two elements which contains (δ, δ), it follows that $\{(\varepsilon, \delta) | \varepsilon \text{ ordinal with } \delta \leqslant \varepsilon < \alpha\}$ is left setwise invariant under G. Then also $T = \{(\varepsilon, \delta) | \varepsilon, \delta \text{ ordinals with } \delta < \beta \text{ and } \delta \leqslant \varepsilon < \alpha\}$ is setwise invariant under G. Now $\{(\beta, \gamma) | \gamma \text{ ordinal with } \gamma \leqslant \beta\}$ is the unique equivalence of class of s which has exactly one element not contained in T (namely (β, β)), hence (β, β) is left fixed under G, and $\{(\beta, \gamma) | \gamma$ ordinal with $\gamma \leqslant \beta\}$ is left setwise invariant under G, hence also pointwise invariant (as each set $\{(\varepsilon, \delta) | \varepsilon$ ordinal with $\delta \leqslant \varepsilon < \alpha\}$ is setwise invariant under G).

Let us finlly consider the case that $\beta = \phi + 1$ and $\alpha = \phi + 2$. We know that r has non-trivial classes $\{(0, 0), (\beta, \phi)\}$, $\{(\phi, \phi), (\beta, \beta)\}$. As there is no other relation of the same type as r, it must be left invariant under G. Then as $(0, 0)$ and (ϕ, ϕ) are fixed by G, we also get (β, ϕ) and (β, β) fixed. Then again the set $\{(\beta, \gamma) | \gamma$ ordinal with $\gamma \leqslant \beta\}$ is setwise fixed by G, and, as above, we can easily see that then also (β, γ) is pointwise fixed by G for all ordinals $\gamma \leqslant \beta$, which completes the induction.

Now we can do a similar construction with semi-automorphisms as we did in the preceding section with automorphisms.

THEOREM 4.3. *Let G be any group. Then there exists an equivalence structure (X, E) such that $G \cong \mathrm{Saut}(X, E)$. If G is finite, we can take X to be finite. If G is infinite, we can take X and E to be of the same cardinality as G.*

PROOF. The theorem is trivial for $|G| \leqslant 2$. So let us assume that $|G| \geqslant 3$. We first take an ordinal α with $\alpha > 2$ such that there exists an injection $j: G \to S(\alpha)$, with the set $S(\alpha)$ as in Example 4.2. Note that we can take α finite if G is finite and of the same cardinality as G if G is infinite. Now let $X = \{0, 1\} \times G \cup \{\infty\} \cup S(\alpha)$. We then define equivalence relations $r, s, t_\gamma, u(\{x, y\})$ with the same equivalence classes as in Example 4.2, having one-element classes on $X \backslash S(\alpha)$. We then define a relation e as having classes $\{\infty\} \cup \{(0, g) | g \in G\}$, $\{(1, g) | g \in G\}$, and one element classes otherwise. For each pair $(g, h) \in G \times G$ we define a relation $\mathrm{f}_{(g,h)}$ as having the class $\{j(h), (0, g), (1, hg)\}$ and one-element classes otherwise. Finally, for every two-element subset $\{x_1, x_2\}$ of X such that x_1, x_2 are not related in any one of the relations defined so far, we define a relation $d(\{x_1, x_2\})$ as having the class $\{x_1, x_2\}$ and one-element classes otherwise. If the set E consists of all relations $r, s, t_\gamma, u(\{x, y\})$, e, $\mathrm{f}_{(g,h)}, d(\{x_1, x_2\})$ then (X, E) is an equivalence structure with X, E of the desired cardinality.

It is not hard to see that there is a natural monomorphism $\phi: G \to \mathrm{Saut}(X, E)$. In fact, if $h \in G$ then the action of $h\phi$ is given by $\infty . (h\phi) = \infty$; $(i, g) . (h\phi) = (i, gh)$ for $i \in \{0, 1\}$, $g \in G$; $x . (h\phi) = x$ for all $x \in S(\alpha)$. So all that remains to prove is that ϕ is surjective.

Let $\beta \in \mathrm{Saut}(X, E)$. Note that e is the only relation in E which contains exactly two non-trivial equivalence classes each having more than two elements. So β has to leave the set $\{0, 1\} \times G \cup \{\infty\}$ setwise invariant. But then β also has to leave its complement, which is $S(\alpha)$, setwise invariant. As in the example it follows that β has to leave $S(\alpha)$ pointwise invariant. Now ∞ is the unique element of $\{0, 1\} \times G \cup \{\infty\}$ which does not lie in any equivalence class having exactly three elements. Hence β has to leave ∞ fixed. Therefore, also β has to leave the sets $\{(0, g)\} | g \in G\}$ and $\{(1, g) | g \in G\}$ setwise invariant.

We denote the unit element of G by i. Let $k \in G$ be such that $(0, i)\beta = (0, k)$. Now if $h \in G$ then we have $j(h)\ \mathrm{f}_{(1,h)}\ (0, i)$ and $j(h)\ \mathrm{f}_{(1,h)}\ (1, h)$. But as $j(h)\beta = j(h)$, we have $j(h)\ \mathrm{f}_{(k,h)}\ (0, k)$, that is, $j(h)\beta\ \mathrm{f}_{(k,h)}\ (0, i)\beta$. We also have $j(h)\ \mathrm{f}_{(k,h)}\ (1, hk)$. Therefore we get $(1, h)\beta = (1, hk) = (1, h)(k\phi)$. In particular, we have $(1, i)\beta = (1, k)$. Now if $h \in G$ we have $j(h^{-1})\ \mathrm{f}_{(h,h^{-1})}\ (0, h)$ and $j(h^{-1})\ \mathrm{f}_{(h,h^{-1})}\ (1, i)$. As above, we have $j(h^{-1})\ \mathrm{f}_{(hk,h^{-1})}\ (1, k)$, that is, $j(h^{-1})\beta\ \mathrm{f}_{(hk,h^{-1})}\ (1, i)\beta$. We also have $j(h^{-1})\ \mathrm{f}_{(hk,h^{-1})}\ (0, hk)$, hence $(0, h)\beta = (0, hk) = (0, h)(k\phi)$. So we have shown that $\beta = k\phi$, which concludes the proof.

REFERENCES

1. R. A. Bailey, C. E. Praeger, C. A. Rowley and T. P. Speed, Generalized wreath products of permutation groups, *Proc. London Math. Soc.* (3), **47** (1983), 69–82.

2. G. Birkhoff, Sobre los grupos de automorfismos, *Revista Unión Mat. Argentina* **11** (1946), 155–157.
3. P. Dembowski, *Finite Geometries*, Springer-Verlag, Berlin, 1968.
4. R. Frucht, Herstellung von Graphen mit vorgegebener abstrakter Gruppe, *Compositio Math.* **6** (1939), 239–250.
5. R. Frucht, Graphs of degree three with a given abstract group, *Canadian J. Math.* **1** (1949), 365–370.
6. R. Frucht, On the construction of partially ordered systems with a given group of automorphisms, *Amer. J. Math.* **72** (1950), 195–199.
7. K. Kuratowski and A. Mostowski, *Set Theory with an Introduction to Descriptive Set Theory*, North Holland, Amsterdam, 1976.
8. E. Mendelsohn, Every group is the collineation group of some projective plane, *J. Geometry* **2** (1972), 97–106.
9. Eric Mendelsohn, On the groups of automorphisms of Steiner triple and quadruple systems, in: *Proceedings of the Conference on Algebraic Aspects of Combinatorics*, University of Toronto, Toronto, Ontario, 1975, *Congressus Numerantium XIII, Utilitas Math.*, Winnipeg, 1975, pp. 255–264.
10. H. Rohrbach, Jürgen Weis, Zum finiten Fall des Bertrandschen Postulats, *J. reine angew. Math.* **214/215** (1964), 432–440.

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